2 Exponential & Logarithmic Functions

2.1 Exponential functions

Rules of exponents:

- $a^{m+n} = a^m \cdot a^n$
- $a^{m-n} = \dfrac{a^m}{a^n}$
- $(a^m)^n = a^{mn}$
- $(a \cdot b)^m = a^m \cdot b^m$
- $\left(\dfrac{a}{b}\right)^m = \dfrac{a^m}{a^n}$

For a fixed number a, where $a > 0$ and $a \neq 1$, define the value of a^x properly, as the range of x extends from integers to rational numbers, and then to real numbers. This gives the idea of the **exponential function**.

Sketch the graphs of $y = a^x$ for both $0 < a < 1$ and $a > 1$, stating the domain and range for each case, and noting the difference in monotonicity.

Exercise 12.

1. State the natural domain and range of the function $y = (0.14)^{\frac{1}{x}}$.
2. Sketch the graphs of $f(x) = -3^{x-1} + 1$, $g(x) = 2^{1-2x}$ and $h(x) = \left|\left(\frac{1}{2}\right)^x - 2\right|$.
3. Compare the following pairs of values:
   (a) $3^{2.8}$ and $3^{2.7}$
   (b) $0.7^{-0.1}$ and $0.7^{-0.2}$
   (c) $1.23^{-1.1}$ and $1.23^{-0.9}$
   (d) 0.6^{-3} and 0.6^{-4}
4. Given that $0.2^m > 0.2^n$, compare the values of m and n.
5. For different possible values of a, solve the inequality $a^{3x-5} > a^{4x+1}$.
6. Determine whether the equation $\left(a^b\right)^c = a^{(b^c)}$ holds in general.
7. Given that the maximum value of the function $f(x) = a^{2x} + 2a^x - 1$, where $a > 0$ and $a \neq 1$, on the interval $-1 \leq x \leq 1$ is 14, find the value of a.
8. (‡) If a non-zero continuous function $f(x)$ defined on $\mathbb{R}$ has the property $f(x+y) = f(x)f(y)$ for all real numbers x and y, show that $f(x) = a^x$ for some $a > 0$.

2.2 Logarithms

For $a > 0$ and $a \neq 1$, if $y = a^x$, then we define the **logarithm to base** a as $x = \log_a y$.

Rules of logarithms include:

- $\log_a 1 = 0$; $\quad \log_a a = 1$; $\quad \log_a a^x = a^{\log_a x} = x$
- $\log_a(xy) = \log_a x + \log_a y$; $\quad \log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$
- $\log_a x^n = n\log_a x$; $\quad \log_{a^n} x = \dfrac{1}{n}\log_a x$; $\quad \log_{a^m} x^n = \dfrac{n}{m}\log_a x$
- $\log_a b = \dfrac{1}{\log_b a}$; $\quad \log_a b = \dfrac{\log_c b}{\log_c a}$

You may wish to write respective rules of exponents for the above equations.

Special bases: $\lg A = \log_{10} A$, and $\ln B = \log_{\mathrm{e}} B$, where $\mathrm{e} = 2.718\,281\,828\,459\,045\ldots$ is a special irrational number.

Exercise 13.

1. Simplify the following expressions:

$$\log_5 125; \qquad \log_6\left(\tfrac{1}{216}\right); \qquad \log_{\frac{1}{4}} 8\sqrt{2}; \qquad \log_2\left(4^7 \times \sqrt{8^3}\right); \qquad (\log_4 3 + \log_8 3)(\log_3 2 + \log_9 2)$$

2. Find the value of xy given that $\log_{11} x = 2$ and $\log_y 27 = -3$.
3. Find the root of the equation $8^{2+x} = 3 \times 9^x$, giving your answer exactly in terms of logarithms.
4. Solve the equation $3^x = 2^{1-2x}$, giving your answer to three decimal places.
5. Solve the simultaneous equations:

$$\begin{cases} 3^x = 5^y \\ x + y = 1 \end{cases},$$

giving your answers exactly in terms of logarithms.

6. Express $\log_{10}\left(5\sqrt{10}\right) - \frac{1}{3}\log_{10} 2.7 - \log_{10}\left(\frac{10}{9}\right)$ in the form of $c + \log_{10} d$, where c and d are rational numbers.
7. Prove the following identities:
   (a) $a^{\log_c b} = b^{\log_c a}$
   (b) $\log_a\left(\frac{u}{v}\right) + \log_a\left(\frac{v}{w}\right) + \log_a\left(\frac{w}{u}\right) = 0.$
   (c) $\log_a b \log_c d = \log_a d \log_c b.$
8. Solve the following equations
   (a) $\log_2(x+2) + \log_2 x = 3$
   (b) $\sqrt{3\log_5 x - 2} - 3\log_5 x + 4 = 0$
   (c) $3 \cdot 4^x - 10 \cdot 2^x + 3 = 0$
   (d) $7^{-x} + 7^{|x|} = 4$
   (e) $\ln(x-3) + 2\ln(x+1) = 3\ln(x-1)$
   (f) $\log_{\frac{1}{2}}(9^x + 1) + \log_{\sqrt{2}}(3^x - 2) + 1 = 0$
   (g) $\dfrac{\log_3 x + 2}{\log_9 x - 2} = \log_{\sqrt{3}} x - 4$
   (h) (†) $\quad \log_x 2 \cdot \log_x 3 + 6 = \log_x 72$

2.3 Logarithmic functions

Sketch the graphs of $y = \log_a x$ in two cases: $0 < a < 1$ and $a > 1$, stating the domain and range for each case, and noting the difference in monotonicity.

Exercise 14.

1. Compare the following pairs of values:
   (a) $\log_7 0.6$ and $\log_7 0.8$
   (b) $\log_{0.4} 4$ and $\log_{0.4} 5$
   (c) $\log_{\frac{3}{5}} 1.4$ and $\log_{\frac{3}{5}} 1.5$
   (d) $\log_{2.1} 0.9$ and $\log_{2.2} 0.9$
   (e) $\log_{\frac{4}{5}} \pi$ and $\log_{\frac{5}{6}} \pi$
2. Sketch the graphs of:
   (a) $f_1(x) = \log_{\frac{1}{4}}(2 - x)$
   (b) $f_2(x) = 2 - \log_{\frac{1}{4}} x$
   (c) $g_1(x) = |\log_3(4x + 1)|$
   (d) $g_2(x) = \log_3 |4x + 1|$
   (e) $g_3(x) = \log_3(4|x| + 1)$
3. Given that $\log_{0.2} m < \log_{0.2} n$, compare the values of m and n.
4. Given two functions $f(x) = \log_a(1 + x)$, and $g(x) = \log_a(1 - x)$, where $a > 0$ and $a \neq 1$.
   (a) State the natural domain of $h(x) = f(x) + g(x)$.
   (b) Determine whether h is an even function or an odd, or neither.
5. The function f is given by $f : x \mapsto \log_a(1 + x)$, where $x > -1$. Define its inverse, f^{-1}, in a similar form.
6. Solve the inequality $\log_2 \left(a^{2x} + 2a^x - 2\right) < 0$, where $a > 0$ and $a \neq 1$.
7. (†) Given that the function $f(x) = \log_a(a - ka^x)$, where $a > 0$ and $a \neq 1$, is a **self-inverse** function, find the value of k. When k has this value, solve the inequality $f(x) \geq 1$.
8. (†) Given that the domain of the function $f(x) = \log_3 \dfrac{mx^2 + 8x + n}{x^2 + 1}$ is $\mathbb{R}$, and that its range is $0 \leq x \leq 2$, find the values of m and n.

Exercise 15.

1. A radioactive source has a half-life of 20 years. Initially its mass is 4.5 kg, find
   (a) the remaining mass after 76 years,
   (b) the time taken such that the remaining mass is only 0.05 kg.

2. An experiment was conducted to discover how fast a particular type of tree grows over time. The results are summarized in the table below, where t is measured in years, and h is the height of the tree being measured, in meters.

t	10	20	30	40
h	2.50	3.10	3.45	3.70

   A model is given by $h = a \log_{10} t + b$, where a and b are constants. Find the values of a and b, and estimate by how many years this tree will grow up to the height of 5 meters.

3. The time taken, T seconds, to compile a file with a source code of x lines by a particular programme is modeled as an exponential function: $T = Ac^x$. An experiment was conducted, and the results are summarized in the table below.

x	50	100	150	200
T	0.658	1.082	1.779	2.926

   Find the values of A and c, and estimate the shortest source code such that its compiling time exceeds 1 minute.

4. (†) To nine decimal places, $\log_{10} 2 = 0.301029996$, and $\log_{10} 3 = 0.477121255$.
   (a) Calculate $\log_{10} 5$ and $\log_{10} 6$ to three decimal places.
   (b) Show that $5 \times 10^{47} < 3^{100} < 6 \times 10^{47}$.
   (c) Hence write down the first digit of 3^{100}.
   (d) Find the first digit of each of the following numbers: 2^{1000}; 2^{10000}; and 2^{100000}.

5. (‡) If a continuous function $f(x)$ defined on $\mathbb{R}^+$ has the property $f(xy) = f(x)+f(y)$ for all positive numbers x and y, show that $f(x) = \log_a x$ for some $a > 0$ and $a \neq 1$.